Big-Oh Notation

Informally, we say that an algorithm that operations on input of size n is worst-case $O(f(n))$ (e.g. $O(n^2)$) if for large values of n the algorithm uses no more than a fixed constant times $f(n)$ basic operations. For example, we will show that the BubbleSort algorithm uses no more than $2n(n-1)$ operations; this is certainly no more than $2n^2$, which is a multiple of n^2, so BubbleSort is worst-case $O(n^2)$.

Stylistically, we try to keep the orders as simple as possible. Since there is an arbitrary constant factor $O(n^2)$ is the same as $O(23n^2)$, which in turn is the same as $O(23n^2+5n+6)$. We represent all of these as $O(n^2)$.

There are several symbols like O:

- O(f(n)) (Big-Oh) represents an upper bound -- for large n the algorithm is no worse than a constant times f(n), but it might be better.
- Ω(f(n)) (Big-Omega) represents a lower bound -- for large n the algorithm is no better than a constant times f(n).
- Θ(f(n)) (Big-Theta) represents both an upper and a lower bound.
- o(f(n)) (Little-Oh) is a strict upper bound: O(f(n)) and not Ω(f(n))

This semester we will use Big-Oh almost exclusively.

Here are a few formal definitions for the mathematically inclined:

- We say function T(n) is O(f(n)) if there are constants k and N so that for every n >= N we have T(n) <= k*f(n).

- We say T(n) is Ω(f(n)) if there are constants k and N so that for every n>=N we have T(n) >= k*f(n).

Don’t worry about these for now.

Here is an addition rule: if $T_1(n)$ is $O(f(n))$ and $T_2(n)$ is $O(f(n))$ then $T_1(n) + T_2(n)$ is $O(f(n))$.

For example, $3n^3-2n^2+27$ is $O(n^3)$ because each of its terms is $O(n^3)$.

In general, a polynomial of degree k is $O(n^k)$ because each of its terms is $O(n^k)$.

Note that all logarithms are proportional -- if a and b are any two bases, then

$$\log_a(x) = \log_b(x)*\log_a(b)$$

If we are only talking about orders of growth, it doesn't matter if we interpret "log" as meaning base-2 logs or base-10 logs or natural logs; each is a constant times each of the others.